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Rise Time Calculations of a Single Air Bubble under the Influence of Gravity in a Pool of Water

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Abstract : *Gas-liquid transfer operations are of considerable interest to the process industries. If the gas is brought into contact with the liquid in the form of bubbles, a better mass transfer is ensured. Usually, their size varies widely, from a few micrometers to a few millimetres. In this work, the influence of the initial bubble diameter, the liquid head and surface tension on the rise time calculations was exemplified for an air bubble in water, using the bubble expansion factor. Further, it was shown that smaller air bubbles are highly influenced by the surface tension. In contrast, for air bubbles in liquid head, if the initial size is larger than 0.1 mm the information needed for the rise time calculation is the liquid head only.*

Keywords - *initial bubble diameter, liquid head, rise time, single bubble, surface tension*

I. INTRODUCTION

Gas-liquid transfer operations are quite common in the process industries. For a better mass transfer, the gas is brought into contact with the liquid in the form of bubbles. The usual bubble size varies from few micrometers to few millimetres. The most commonly used gas-liquid contactors are bubble column reactors, mineral processing devices, foaming devices and fermenters. In these contactors, two phases are brought into intimate contact in the form of bubbles. The interfacial surface of the bubble, the nature and degree of dispersion of the one fluid in the other influences the rate of mass transfer. In these contactors, bubbles swarms rather than a single bubble, but the understanding of the single bubble hydrodynamics provides an important insight and often serves as a useful starting point to undertake the modelling applications, involving bubble swarms [1]. For a given type of gas-liquid contactor, the knowledge of free rise bubble velocity provides the estimation of the mean residence time of the gas phase. Once the desired mean residence time is known, the sizing of the gas-liquid equipment can be done. This gives the motivation to study the rise time calculations for bubbles and parameters that affect the bubble's hydrodynamics. Because of this, an intensive research has been done [1]. In the past, several efforts have been made to identify the behaviour of the single gas bubble [2-17]. Previous investigations were mainly focused on the bubble terminal velocity in a steady motion, and the details are summarised in Table 1.

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Table 1. Summary of single air bubbles behaviour obtained from the literature [7, 12, 16, 18]

Sl. No	Region (equivalent diameter, d)	Behaviour of the bubble	Assumptions	Terminal velocity expression
1.	Region1 (d < 0.7 mm)	like an expanding sphere without internal circulations	No slip condition on the tangential velocity[16]	$\frac{gd_p^2(\rho_L-\rho_g)}{18\mu}$
2.	Region 2 (0.7 mm< d <1.4 mm)	like an expanding sphere with internal circulations	No shear stress transmitted across the interface [18]	$\frac{gd_p^2(\rho_L-\rho_g)}{12\mu}$
3.	Region 3 (1.4 mm < d < 6 mm)	No longer like a spherical and tend to follow a zigzag or helical path oscillatory	Based on Wave Theory [7] $\sqrt{\frac{2\sigma g_c}{d_p\rho_L}+\frac{g\,d_p}{2}}$	
4.	Region 4 (d > 6 mm)		Based on Isaac H. Lehrer rational arguments [12] $\sqrt{\frac{3\sigma g_c}{d_p\rho_L}+\frac{\Delta\rho g\,d_p}{2\rho_L}}$	

In this article, calculations for the estimation of the bubble rise time under the influence of gravity in a low viscosity fluid are presented. The bubble expansion factor [18] has been used for analysing the influence of the initial bubble diameter, the liquid head and the surface tension. The introduction of the paper should explain the nature of the problem, previous work, purpose, and the contribution of the paper. The contents of each section may be provided to understand easily about the paper.

II. THEORY

Let us consider a single gas bubble with initial diameter ( ), released in a pool of liquid (density,) under atmospheric pressure , at a depth . As the bubble is released, it will start moving upwards freely and undergoes expansion as it rises to the surface (Fig. 1). This is due to the reduction in the hydrostatic pressure to which it is subjected. The bubble behaviour depends on various parameters such as initial diameter, liquid head and surface tension.

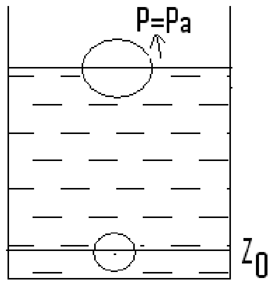

Fig.1. A bubble in a pool of liquid.

III. ANALYSIS

Case I. **Influence of the Surface tension**. Assuming that the air bubble is not dissolving in the water as it rises upwards in the liquid, and has only one surface and the liquid temperature is constant, the following analysis was made. Let the air bubble equivalent diameter (since it may not be spherical) increases from to (where) as it reaches to the top of the liquid column. For isothermal system, according to ideal gas law

PV = constant. (1)

Considering the initial size of the air bubble is d_{po} at a depth Z_o, the pressure in the air bubble at initial state is

$$\underbrace{P_A}_{\substack{Atmospheric\\ pressure}} + \underbrace{\rho L g Z_o}_{\substack{liquidhead\\ pressure}} + \underbrace{4\sigma / d_{po}}_{\substack{Excess\,of\,pressure\\ surfacetensionpressure}} \quad (2)$$

and the Pressure in the air bubble at final state (i.e., at the surface of the pool) is

$$P_A + 4\sigma / \lambda d_{po} \quad (3)$$

where

Initial volume = $\pi d_{po}{}^3 / 6$,

Final volume = $\pi\lambda^3 d_{po}{}^3 / 6$.

According to the Eq. (1) it follows that

$$\left(P_A + \rho_L g Z_O + 4\sigma / d_{po}\right) \pi/6\, d_{po}^3 = \left(P_A + 4\sigma / \lambda d_{po}\right) \pi/6\, \lambda^3 d_{po}^3$$ and respectively

$$Z_o = \frac{P_A d_{po}[\lambda^3 - 1] + 4\sigma[\lambda^2 - 1]}{\rho_L g d_{po}}, \quad (4)$$

where

Initial terminal velocity (i.e. based on initial size) = $\frac{g d_{po}^2 \Delta\rho}{18\mu}$ m/s,

Final terminal velocity (i.e. based on final size) = $\frac{g \lambda^2 d_{po}^2 \Delta\rho}{18\mu}$ m/s,

Average velocity (terminal) = $\frac{g d_{po}^2 \Delta\rho}{18\mu}\left[\frac{\lambda^2+1}{2}\right]$ m/s. (5)

Thus, for the rise time, t_I, the following equation was obtained:

$$t_I = \frac{Z}{\frac{g d_{po}^2 \Delta\rho}{18\mu}\frac{[\lambda^2+1]}{2}} \text{ s.} \tag{6}$$

Let us assume the following possibilities:

a) The surface tension is negligible

If the surface tension is negligible, Eq. (4) is reduced to

$$Z_o = \frac{P_A}{\rho_L g}[\lambda^3 - 1] \tag{7}$$

and $\lambda^3 = 1 + \frac{\rho_L g Z_o}{P_A}$, $\lambda = \left(1 + \frac{\rho_L g Z_o}{P_A}\right)^{1/3}$. (8)

b) With surface tension

In this case Eq. (4) is rewritten as

$$(\lambda)^3 + \frac{4\sigma}{P_A d_{po}}(\lambda)^2 - \left[1 + \frac{\rho_L g Z_o}{P_A} + \frac{4\sigma}{P_A d_{po}}\right] = 0 . \tag{9}$$

Eq. (9) is a cubic equation and has only one real root. More details about solution method are available elsewhere [19].

$$(\lambda)^3 + A(\lambda)^2 - [A+B] = 0 , \tag{10}$$

Where $A = \frac{4\sigma}{P_A d_{po}}$, $B = 1 + \frac{\rho_L g Z_o}{P_A}$,

$$\lambda = \delta - A/3 , \tag{11}$$

$$\delta^3 + p\delta + q = 0 , \tag{12}$$

$$p = \frac{1}{3}(-A) \ \& \ q = \frac{1}{27}(27(A+B) + 2A^3), \tag{13}$$

$$r = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 . \tag{14}$$

Therefore, r always > 0, hence, the cubic equation has only one real root:

$$\delta = \left(-\frac{q}{2} + \sqrt{r}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{r}\right)^{1/3} . \tag{15}$$

Combining Eq. (11) and Eq. (15) we get analytical solution for λ (bubble expansion factor). Using MATLAB® Eq. (8) and Eq. (9) have been solved to obtain Fig. 2 and Fig. 3. These figures show the influence of the initial air bubble diameter, the liquid head, and the surface tension on the bubble expansion. Fig. 2 shows the influence of the surface tension and the liquid head on the bubble expansion for 1 μm air bubble. When surface tension affects are taken into consideration, the expansion of 1 μm bubble is found to be about 10% in 10 m liquid head, whereas for the same liquid head with negligible surface tension, the bubble expansion is about 25%. From Fig. 3 it is clear that when initial air bubble size is 0.1 mm, the expansion of the bubble is about 25% for both cases with and without surface tension. Therefore, we can say that the smaller air bubbles are highly influenced by surface tension. For bubbles larger than 0.1 mm, the expansion is observed to be about 25% in 10 m liquid head and it is independent from the initial size and depends only on the liquid head. From Fig. 3, it is also clear that the influence of the surface tension on the bubble expansion is insignificant. Therefore, for air bubbles in liquid head, if the initial size is larger than 0.1 mm the information needed for the calculation rise time is the liquid head only. Therefore, in the following sections (under Case II & Case III) for the case of 0.5 mm bubbles liquid head information alone is considered to calculate the bubble rise time.

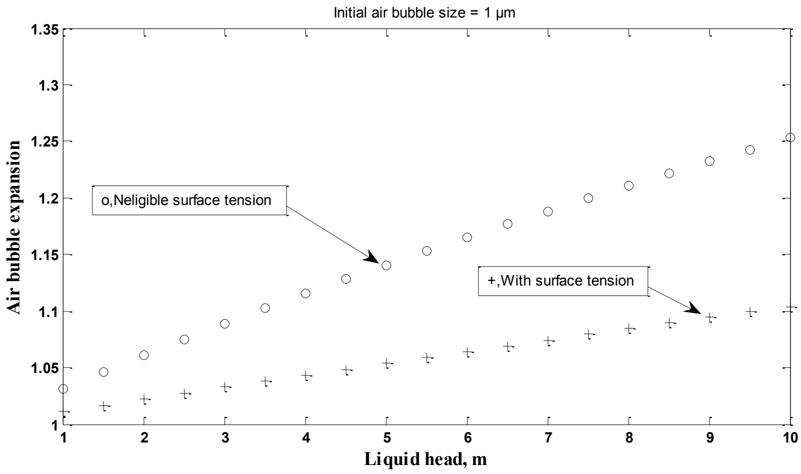

Fig. 2. The air bubble expansion vs. the water liquid head for 1 μm air bubble with and without surface tension.

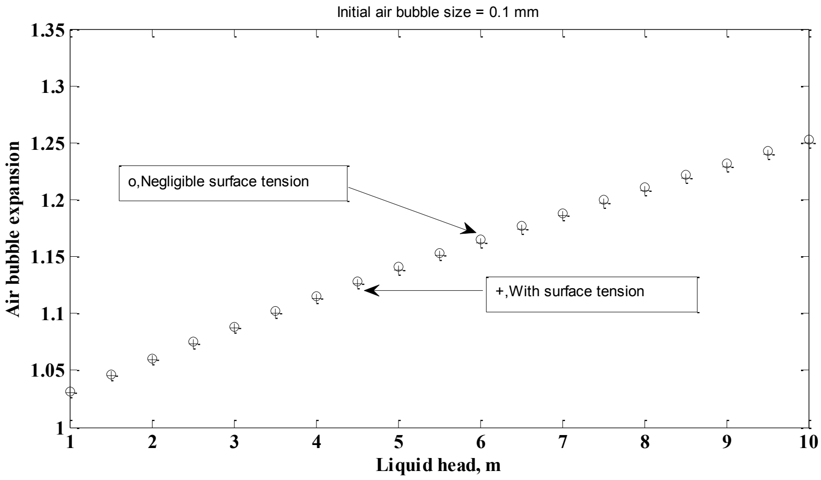

Fig. 3. The air bubble expansion vs. the water liquid head for 0.1 mm air bubble with and without surface tension.

Illustration: Rise time calculations for 5×10^{-4} m bubble in a 10 m water column. According to the information in Table 1, if the air bubble initial diameter (i.e., equivalent diameter, d) is less than 0.7 mm, bubble assumes terminal velocity and it is given by Stoke's law [16]. It is known that the surface tension of the water is $\sigma = 0.072\,{}^{N}\!/_{m}$; the atmospheric pressure $P_A = 101.325\ {}^{\text{KN}}\!/_{\text{m}^2}$; $g = 9.81\dfrac{m}{s^2}$; and ρ_L of the water at 25°C = 1000 ${}^{\text{kg}}\!/_{\text{m}^3}$. It was calculated that λ is 1.25, using Eq. (4). In this case percentage rise in the bubble size = 25.24%.
Density of the air at 25°C is

$$\rho_{air} = {}^{P_A M}\!/_{RT} = \frac{101.325\times10^3\times29}{8314\times298}\ {}^{\text{kg}}\!/_{\text{m}^3} = 1.186\ {}^{\text{kg}}\!/_{\text{m}^3},$$

and rise time, $t_I = \dfrac{10}{\dfrac{9.81(5\times10^{-4})^2(1000-1.186)}{18\times10^{-3}}\left[\dfrac{(1.25238)^2+1}{2}\right]}\ \text{s} = 57.28$ s.

Case II. Considering negligible Surface tension. In the case when the surface tension is neglected, it can be calculated according to Eq. (2). Let the final size be d_f and then the pressure inside the bubble is

$$P_A + 4\sigma/d_f \text{ , } [\rho_L g Z = 0]. \tag{16}$$

Since the mass of the bubble is constant, it follows that

$$\pi d_{po}^3/6 \left(P_A + \rho_L g Z_0 + 4\sigma/d_{po}\right) = \pi d_f^3/6 \left(P_A + 4\sigma/d_f\right). \tag{17}$$

Eq. (17) manipulated as follows

$$\pi d_{po}^3/6 \left(P_A + \rho_L g Z_0 + 4\sigma/d_{po}\right) = \pi d_f^3/6 \left(P_A + 4\sigma/d_{po} \times d_{po}/d_f\right). \tag{18}$$

In Eq. (18) and $4\sigma/d_{po} \times d_{po}/d_f$ is negligible.

For low Reynolds number Drag is given by $3\pi d\mu v$. Since the surface tension is neglected, Eq.(18) reduced as follows:

$$\left(\frac{d_p}{d_{po}}\right)^3 = \frac{(P_A + \rho_L g Z_0)}{(P_A + \rho_L g Z)} \text{ , } d_p = d_{po}\left(\frac{P_A + \rho_L g Z_0}{P_A + \rho_L g Z}\right)^{1/3}. \tag{19}$$

When the single bubble equivalent diameter is less than 0.7 mm it will be in spherical shape and behaves like a rigid sphere [16, 18]. Therefore, the drag force for rigid sphere is $3\pi d_p \mu v$. For the force balance the following equation is valid:

$$0 = \text{Buoyancy force} - \text{Drag force}$$

$$= \frac{\pi d_p^3 \rho_L g}{6} - 3\pi\, d_p\, \mu v \quad ;$$

$$v = \frac{d_p^2}{18}\frac{\rho_L g}{\mu}.$$

The substitution of d_p in terms of d_{po}, here d_p is the diameter at any instant $d_p(t)$ and v is the velocity at any instant $v(t)$

$$\text{implies } -\frac{dZ}{dt} = \frac{d_{po}^2}{18}\frac{\rho_L g}{\mu}\left(\frac{P_A + \rho_L g Z_0}{P_A + \rho_L g Z}\right)^{2/3}, \tag{20}$$

$$-(P_A + \rho_L g Z)^{2/3} dZ = \frac{d_{po}^2}{18}\frac{\rho_L g}{\mu}(P_A + \rho_L g Z_0)^{2/3} dt. \tag{21}$$

Integrating between limits of Z (Z_o to 0) and t (0 to t_{II})

$$-\frac{3}{5}\frac{1}{\rho_L g}(P_A + \rho_L g Z)^{5/3}\Big|_{Z_0}^{0} = \frac{d_{po}^2}{18}\frac{\rho_L g}{\mu}(P_A + \rho_L g Z_0)^{2/3} t\Big|_0^{t_{II}}, \tag{22}$$

$$\frac{3}{5}\frac{1}{\rho_L g}\left[(P_A + \rho_L g Z_0)^{5/3} - p_a^{5/3}\right] = \frac{d_{po}^2}{18}\frac{\rho_L g}{\mu}\left[(P_A + \rho_L g Z_0)^{2/3}\right] t_{II}. \tag{23}$$

$$\text{The rise time, } t_{II} = \frac{54}{5}\frac{\mu}{d_{p0}^2}\frac{1}{\rho_L^2 g^2}\frac{\left((P_A + \rho_L g Z_0)^{5/3} - P_A^{5/3}\right)}{(P_A + \rho_L g Z_0)^{2/3}}. \tag{24}$$

Substituting all parameters in Eq. (24) we get rise time calculations for 5×10^{-4} m bubble in a 10 m water column, $t_{II} = 54.06$ s.

Case III. Considering just the liquid head information. The following manipulation has been suggested for the rise time calculations, where the gas phase properties are considered at average liquid head pressure. For rigid sphere [18] the terminal velocity is given by Stoke's law and is as follows:

$$v_t = \frac{g d_p^2 (\Delta\rho)}{18\mu_L} = \frac{g d_p^2 (\rho_L - \rho_g)}{18\mu_L} = \frac{g d_p^2 \rho_g \left(\rho_L/\rho_g - 1 \right)}{18\mu_L} . \quad (25)$$

Applying conservation of the mass on the air bubble gives

$$\underbrace{\frac{\pi}{6} d_{po}^3}_{Initial volume} \rho_{go} = \underbrace{\frac{\pi}{6} d_{p,ave}^3}_{Average volume} \rho_{g,ave} . \quad (26)$$

After rearrangement the Eq.(26) transforms in

$$d_{p,ave} = d_{po} \left(\rho_{go}/\rho_{g,ave} \right)^{1/3} . \quad (27)$$

This implies $d_{p,ave}^2 \rho_{g,ave} = d_{po}^2 (\rho_{go})^{2/3} (\rho_{g,ave})^{1/3}$, (28)

where ρ_{go} is the initial density of the air bubble, $\rho_{g,ave}$ is the average density of the air bubble, d_{po} is the initial diameter of the air bubble, and $d_{p,ave}$ is the average diameter of the bubble.

Therefore,

$$v_t = \frac{g d_{po}^2 \rho_{go}^{2/3} \rho_{g,ave}^{1/3} \left(\rho_L/\rho_{g,ave} - 1 \right)}{18\mu_L} . \quad (29)$$

We know that $\text{velocity} = \frac{\text{distance}}{\text{time}}$ and as a result,

$$\frac{Z}{t_{III}} = \frac{g d_{po}^2 \rho_{go}^{2/3} \rho_{g,ave}^{1/3} \left(\rho_L/\rho_{g,ave} - 1 \right)}{18\mu_L} . \quad (30)$$

Consequently, the rise time, $t_{III} = \frac{18\mu_L Z}{g d_{po}^2 \rho_{go}^{2/3} \rho_{g,ave}^{1/3} \left(\rho_L/\rho_{g,ave} - 1 \right)}$. (31)

Substituting all parameters in Eq. (31) we get rise time calculations for 5×10^{-4} m bubble in a 10 m water column.
The initial pressure on the bubble is computed as follows

$$\left(\underbrace{101.325}_{Atomspheric\ pressure} + \underbrace{10 \times 1000 \times 9.81/1000}_{Pressure\ due\ to\ liquid\ column} \right) KN/m^2 = 199.4\ KN/m^2 .$$

The average pressure of the liquid column is computed as follows

$$\left(\underbrace{101.325}_{Atomspheric\ pressure} + \underbrace{5 \times 1000 \times 9.81/1000}_{Average\ pressure\ in\ the\ column} \right) KN/m^2 = 150.35\ KN/m^2 .$$

Air bubble density at initial conditions and 25°C = $\frac{29}{22.414} \times \frac{273}{298} \times \frac{199.4}{101.325} = 2.33\ kg/m^3$.

Air bubble density at average pressure and 25°C = $\frac{29}{22.414} \times \frac{273}{298} \times \frac{150.35}{101.325} = 1.76\ kg/m^3$.

Therefore, the rise time is calculated as,

$$t_{III} = \frac{18\times 0.8937\times 10^{-3}\times 10}{9.81\times 0.5^2\times 10^{-6}(2.33)^{2/3}(1.76)^{1/3}\left[\left(1000/1.76\right)-1\right]} = 54.5 \text{ s.}$$

All the three cases discussed in this study resulted in almost equal rise time for 0.5 mm air bubble. However, case two and three are very close. This is due the fact that integration is nothing but differential summation. However, it is very easy to use average properties while calculating rather than using integration.

IV. CONCLUSION

The influence of the initial bubble size, the liquid head and the surface tension on the bubble expansion has been presented analytically. The MATLAB code that has been given can be used to demonstrate the influence of these parameters. All the three cases discussed in this study resulted in approximately equal rise time for an air bubble. Although, the rise time calculations are exemplified with 0.5 mm air bubble, a similar approach can be used for bubbles with other size. It is important to note that these time calculations are appropriate for non-dissolving bubbles.

NOTATION

A–Dimensionless parameter ($A = 4\sigma / P_A d_{po}$);
B–Dimensionless parameter ($B = 1 + \rho_L g Z_o / P_A$);
d–Equivalent diameter of the gas bubble, m;
g–Acceleration due to gravity, m/s^2;
M-Molecular weight of air;
P–Pressure, N/m^2;
p–Number in terms of *A* ($p = 1/3(-A)$);
q–Number in terms of *A*&*B* ($q = \frac{1}{27}(27(A+B)+2A^3)$);
R–Universal gas constant;
r–Number in terms of *p* & *q* ($r = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$);
t–Time, s;
V–Volume (for sphere $\frac{\pi}{6}d^3$), m^3;
v–Velocity, m/s;
Z–Distance, m;
μ–Viscosity, kg/ms;
σ-Surface tension, N/m;
ρ–Density, kg/m^3;
λ–Constant (greater than 1);
Δ–Difference.

SUBSCRIPTS

A–Atmospheric;
air–Air;
ave–Average liquid head conditions;
f–Final state;
g–Gas;
L–Liquid;
p_o–Initial condition ($t = 0$);
p–Instant equivalent diameter of the bubble ($t > 0$);
t–Terminal;
I–Case I;

II–Case II;
III–Case III.

APPENDIX

1. Matlab m-files for bubble expansion vs. liquid head

```
% Program to calculate air bubble expansion in water column
% Air bubble expansion with negligible "σ"
% Air bubble expansion with "σ"
clear all ;
% data
T  = 0.072 ; % surface tension N/m
mu = 0.8937*10^-3 ; % viscosity Kg/m s
ro = 1000 ; % density Kg/m^3
pa = 1.01325*10^5 ; % atmospheric pressure N/m^2
g  = 9.81 ; % acceleration due to gravity m/s^2
count = 1 ;
z=[1;1.5;2;2.5;3;3.5;4;4.5;5;5.5;6;6.5;7;7.5;8;8.5;9;9.5;10];% head m
d0 = 1*10^-6; % initial size m
%d0 = 1*10^-4; % note for initial size 0.1 mm
for j=1:1:length(z)
   H=z(j)
   alpha=(1+ro*g*H/pa)^(1/3)
   ps = ro*g*H ; % hydrostatic column pressure
   a = 1 + (pa/ps);
   b = (4*T)/(ps*d0) ;
   %########################################################################
   p = (-1/3)*(b/(a-1))^2 ;
   q = -1*(a+b)/(a-1) + (2/27)*(b/(a-1))^3 ;
   r = (p/3)^3 + (q/2)^2
   a1 = ((-q/2) + sqrt(r))^(1/3) ; b1 = ((-q/2) - sqrt(r))^(1/3) ;
   alphaNN = a1+b1 - (1/3)*(b/(a-1))
%########################################################################
   alphaN(count)=alpha % Air bubble expansion with negligible "σ"
   alphaNNN(count)=alphaNN % Air bubble expansion with "σ"
   count = count + 1 ;
end ;
plot(z,alphaN,'o')
hold on
plot(z,alphaNNN,'+')
xlable(' Liquid head in m')
ylable('Air bubble expansion')
```

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